

THE PRINCIPAL TYPE-SCHEME OF AN OBJECT IN COMBINATORY LOGIC

BY
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Introduction. In their book *Combinatory Logic* [1], Curry and Feys introduced the notion of “functional character” (here called “type-scheme”) of an object of combinatory logic. Roughly speaking, each object of combinatory logic (“ob” for short) represents a function or an operator on functions; for instance the ob **I** represents the identity operator, and we have for all obs X ,

$$\mathbf{I}X = X.$$

One of the aims of combinatory logic is to study the most basic properties of functions and other concepts, with as few restrictions as possible; hence in the simplest form of combinatory logic there is nothing to stop a function being applied to itself; thus XX is an ob for all obs X . However it is also natural to look at the consequences of introducing type-restrictions, formulated by assigning types to the obs according to certain rules, to be described later.

Each type is a linguistic expression representing a set of functions, and for any type α the statement “ X has type α ” is intended to mean that the ob X represents a member of the set represented by α . Given types α and β , the set of all functions from the set α into the set β is represented by the type “ $\mathbf{F}\alpha\beta$ ” (using Curry’s notation). Now consider the ob **I**; if X has type α , then the ob $\mathbf{I}X$ must also have type α . Hence **I** represents a function from α into α , and so it must be given the type $\mathbf{F}\alpha\alpha$, for each type α . Thus **I** has not just one type, but a whole class of types. This might seem strange, but it comes from the fact that **I** represents the abstract notion of “identity-function”, rather than one particular identity-function for a particular set. The identity-function involves basically the same concept, no matter what we are applying it to. Similarly the other two basic combinators **S** and **K** (see later, or [1, §5A]) represent certain simple operations which can be performed on almost any functions, and thus they too have an infinite class of types (see §1 later for details, and §5 for comment).

To denote classes of types, we can use variables a, b, c ; then the fact that **I** has every type of form $\mathbf{F}\alpha\alpha$ can be expressed by assigning to **I** the *type-scheme*

$$\mathbf{F}aa.$$

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We shall see later that all the types which the rules assign to \mathbf{I} have the form $\mathbf{F}\alpha\alpha$, so the types of \mathbf{I} are just those obtained by substituting a type α for the variables in the type-scheme $\mathbf{F}\alpha\alpha$. It will not be obvious from the rules that every ob X has a type-scheme with this property. If X does have a type-scheme from which all the types of X (and no extra types) can be obtained by substituting types for variables, it will be called a *principal type-scheme* of X ⁽¹⁾.

In §3 of this paper it will be proved that every ob X that has a type at all, has in fact a principal type-scheme which can be effectively determined from the structure of the ob. This result will apply to the case when X may contain variables, which are not assigned types by the rules; in this case it will say that if there exists an assignment of types to the variables from which a type for X can be deduced, then there is a type-scheme α such that all the types of X , for every assignment of types to the variables, are instances of α .

§4 will show how the principal type-scheme of $[x]. X$ (defined therein and in [1]) is related to the principal type-scheme of X .

Finally, in §6 it will be shown that if α is a principal type-scheme of an ob X , then any substitution-instance β of α is a principal type-scheme of some ob X_β (which is reducible to X by the reduction-rules for combinators). Besides its intrinsic interest, this result goes part of the way towards justifying a conjecture of Curry's that the alternative system of combinatory logic with type-restrictions (in which an infinity of basic combinators is postulated, each with a unique type) can be defined in the system described here. This point will be explained in §5.

Actually, the main result (Theorem 1) in §3 was proved independently, almost simultaneously, by Curry and myself, using different methods; the present exposition owes something to both. Curry's proof (see [2]) does not use the result of J. A. Robinson [4] which is needed here; in fact I think his proof essentially contains an alternative proof of that result. The main result in §4 (Theorem 2) was also proved almost simultaneously by Curry and myself: since Curry's proof is shorter, it has been used here.

I wish to thank H. Curry for all his help with this paper, both in discussions and by permission to use his proof of Theorem 2.

1. Definitions. The notions presented informally in the introduction will now be formalized. The symbol " \equiv " will be used to denote identity of obs, types and type-schemes, since " $=$ " will denote a defined equivalence relation (roughly, equality of interpretation) between obs. Obs will be regarded as linguistic expressions

⁽¹⁾ In [2] principal type-schemes are called "principal functional characters." In [1] the term "functional character" is used in approximately the sense of "principal type-scheme" here, but the assumptions therein allow that extra types could be assigned to the obs, besides the types given explicitly; if this were done the functional characters stated in [1] (e.g. §8c, p. 264) would not be principal.

intended to represent functions or more abstract operations (like **I** in the Introduction)⁽²⁾.

DEFINITION 1. Obs.

(i) There are assumed to be certain symbols called *ob-atoms*, all of which are obs; these atoms may include two *basic combinators*, **S** and **K**.

(ii) If X and Y are obs, then the result, (XY) , of putting X and Y in parentheses with X on the left of Y , is a *compound ob*.

A *combinator* is any ob in which only the atoms **S** and **K** occur.

(Notice that this definition is really a sort of definition-scheme: for each given set of atoms, it defines by induction a corresponding set of obs. The exact nature of the atoms is irrelevant here (except that **S** and **K** will play a special role). Similarly Definitions 2, 4 and 5 are really definition-schemes.)

It is assumed that no compound ob is also an atom, and that if $(XY) \equiv (UV)$, then $X \equiv Y$ and $U \equiv V$. Capital Roman letters denote arbitrary obs, and parentheses will be omitted in such a way that for instance, " $XYZU$ " denotes $((XY)Z)U$.

DEFINITION 2. *Types and Type-schemes.*

(i) There are assumed to be certain symbols called *basic types* (at least one and at most countably many); also a countably infinite sequence of *type-variables*. All these are type-schemes, called *type-atoms*.

(ii) If α and β are type-schemes, then $(F\alpha\beta)$ is a *compound type-scheme*.

A *type* is any type-scheme which contains no variables⁽³⁾.

It is assumed that no compound type-scheme is a type-atom and that if $(F\alpha\beta) \equiv (F\gamma\delta)$ then $\alpha \equiv \beta$ and $\gamma \equiv \delta$. Greek letters denote arbitrary type-schemes, and lower case Roman letters from the start of the alphabet denote type-variables. The outermost pair of parentheses on a type-scheme will usually be omitted. In denoting lists of variables, for example " a_1, \dots, a_n ", it will be assumed that the variables are mutually distinct (more precisely, $i \neq j$ implies $a_i \neq a_j$), unless explicitly stated otherwise.

In the usual interpretations, the ob (XY) represents the result of applying the function X to the argument Y . Each basic type represents a particular set of functions, depending on the interpretation one may have in mind, and $F\alpha\beta$ represents the set of all functions from α into β . Thus each type represents a particular set of functions.

The variables are intended to represent arbitrary types, so that a type-scheme α containing variables represents a class of types, each type being obtained from α by substituting types for the variables.

⁽²⁾ In Curry [1] the obs are not necessarily linguistic expressions, but it is convenient to think of them here as such, and this specialization involves no essential loss of generality. But in Definition 1, notice for example that the symbols "**S**" and "**K**" need not themselves be the basic combinators; they only denote them.

⁽³⁾ In [1], Curry finds it convenient to regard types as a particular kind of ob, called "*F-obs*"; the results of the present paper apply to this case.

Sequences will be denoted by angle brackets, for example “ $\langle \alpha_1, \dots, \alpha_n \rangle$ ”, and sets denoted by brackets “ $\{ \}$ ”.

DEFINITION 3. *Substitution.* For any type-schemes β_1, \dots, β_n , α , and distinct type-variables a_1, \dots, a_n ,

$$\left[\frac{\beta_1, \dots, \beta_n}{a_1, \dots, a_n} \right] \alpha$$

is defined to be the result of replacing each occurrence of a_i in α by β_i , simultaneously for $i = 1 \dots n$. (If a_i does not occur in α , no replacements are made.) Any type-scheme with the above form is called an *instance* of α . (Hence α is an instance of itself.) For any sequence $\langle \alpha_1, \dots, \alpha_k \rangle$ of type-schemes, define

$$\left[\frac{\beta_1, \dots, \beta_n}{a_1, \dots, a_n} \right] \langle \alpha_1, \dots, \alpha_k \rangle$$

to be the sequence formed by substituting β_i for a_i in all of $\alpha_1, \dots, \alpha_k$, simultaneously for $i = 1, \dots, n$.

An expression consisting of a type-scheme α beside an ob X is called a *statement*, αX . The statement βX is called an *instance* of αX iff β is an instance of α .

DEFINITION 4. *Type-schemes of obs.* The type-scheme α is a *type-scheme of X* iff the statement αX can be deduced from the axioms below by the rule (F) below.

Axiom-schemes. Each axiom-scheme is assumed to be a statement βY , where Y is an ob-atom, and no two axiom-schemes contain the same Y . If the ob-atoms include **S** and **K**, then the axiom-schemes include

$$\begin{aligned} & \mathbf{F(Fa(Fbc))(F(Fab)(Fac))S} \\ & \mathbf{Fa(Fba)K} \end{aligned}$$

for some (distinct) type-variables a, b, c . The *axioms* are all the instances of the axiom-schemes. (Including the schemes themselves; this is just a notational convenience.)

Rule (F). From $(F\gamma\delta)X$ and γY , deduce $\delta(XY)$.

If α contains no variables, and αX is deduced from axioms containing no variables, we say that α is a *type of X* .

DEFINITION 5. *Deductions.* Given a set \mathcal{A} of statements, a *deduction* of αX from \mathcal{A} is a finite set of statements arranged as a tree in the usual way, with αX at the bottom. Each statement in the tree must either have just two statements immediately above it, and follow from them by Rule (F), or else be an axiom or member of \mathcal{A} , with nothing above it. The expression

$$\mathcal{A} \vdash \alpha X$$

means that there exists a deduction of αX from \mathcal{A} ; and

$$\vdash \alpha X$$

means that there is a deduction of αX whose branch-tops are all axioms⁽⁴⁾.

⁽⁴⁾ We shall only be interested in deductions in which each ob in \mathcal{A} is an atom not occurring in the axiom-schemes. Throughout this paper “deduction” means this kind of deduction.

The axiom-schemes for **S** and **K** in Definition 4 are justified in [1, p. 263], by their connection with the reduction-rules for these two combinators (see (13) later), but for the present section these reduction-rules are irrelevant. The restrictions on the unspecified axiom-schemes in Definition 4 will ensure that no ob-atom has two different principal type-schemes. Rule (F) can be interpreted as saying that if X is a function from the set γ into the set δ , and Y is in γ , then (XY) is in δ ⁽⁵⁾.

Incidentally, sometimes the ob **I** of the Introduction is included as an extra basic combinator, and sometimes it is defined in terms of **S** and **K**; the present paper includes both possibilities. If **I** was included as an ob-atom, one of the unspecified axiom-schemes of Definition 4 would be $(Faa)\mathbf{I}$. Some theories based on combinatory logic postulate extra axioms $(F\alpha\beta)\mathbf{I}$ for certain distinct types α and β (meaning intuitively that α is a subset of β). In this case **I** would not have a principal type-scheme. In fact if **I** was an atom the condition in Definition 4 that no two axiom-schemes contain the same Y would not be satisfied, and if **I** was compound, there would be an axiom with a compound ob; thus the present paper would not apply.

From Definitions 4 and 5 it can be seen that if the obs in the statements in \mathcal{A} are atoms, then the steps in a deduction

$$\mathcal{A} \vdash \alpha X$$

follow the construction of X ; if X is an atom, the deduction must consist of one step only, while if X is (UV) , the deduction must be the result of joining together two deductions with the form

$$\mathcal{A} \vdash F\beta\alpha U, \quad \mathcal{A} \vdash \beta V$$

by an application of Rule (F). Hence every ob-atom in X must occur in a statement in \mathcal{A} or in an axiom; and each statement in \mathcal{A} which actually occurs in the deduction must have its ob-atom occurring in X [1, §9B, Theorem 1].

EXAMPLE 1. $\vdash Faa(\mathbf{SKK})$.

Proof.

$$\left\{ \begin{array}{l} \frac{F(Fa(F(Fba)a))(F(Fa(Fba))(Faa))\mathbf{S} \quad Fa(F(Fba)a)\mathbf{K}}{F(Fa(Fba))(Faa)(\mathbf{SK}) \quad Fa(Fba)\mathbf{K}} \\ \hline Faa((\mathbf{SK})\mathbf{K}) \end{array} \right.$$

The first two statements in this deduction, for **S** and **K**, are axioms obtained by substituting Fba for b , and a for c , in the appropriate axiom-schemes.

(⁵) If **S** and **K** are the only ob-atoms, and type-variables are prohibited from the axioms, Definition 4 is equivalent to the system in [2], and to the basic theory $\mathcal{F}_1(\mathbf{K}, \mathbf{S})$ in [1, §9A3] with restricted rule F and rule Eq omitted.

DEFINITION 6. *Instances of deductions.* For any deduction \mathcal{D} , any type-schemes β_1, \dots, β_n , and (distinct) type-variables a_1, \dots, a_n ,

$$\left[\frac{\beta_1, \dots, \beta_n}{a_1, \dots, a_n} \right] \mathcal{D}$$

is defined to be the result of simultaneously substituting β_1, \dots, β_n for a_1, \dots, a_n in all the type-schemes in \mathcal{D} , and it is called an *instance* of \mathcal{D} .

If the above substitution is denoted by “*”, it can be seen that if \mathcal{D} is a deduction giving

$$\{\alpha_1 Y_1, \dots, \alpha_k Y_k\} \vdash \beta X,$$

then \mathcal{D}^* will be a deduction giving

$$\{\alpha_1^* Y_1, \dots, \alpha_k^* Y_k\} \vdash \beta^* X.$$

DEFINITION 7. *Principal type-schemes.*

(a) Suppose each atom in X occurs in an axiom-scheme; then a type-scheme α is a *principal type-scheme* (p.t.s.) of X iff

- (i) $\vdash \alpha X$, and
- (ii) if $\vdash \beta X$, then β is an instance of α .

(b) If X contains ob-atoms x_1, \dots, x_n (mutually distinct) which are not given types by the axiom-schemes, then α is said to be a p.t.s. of X iff

- (i) for some type-schemes $\alpha_1, \dots, \alpha_n$,

$$\{\alpha_1 x_1, \dots, \alpha_n x_n\} \vdash \alpha X,$$

- (ii) and if for some type-schemes $\beta_1, \dots, \beta_n, \beta$,

$$\{\beta_1 x_1, \dots, \beta_n x_n\} \vdash \beta X,$$

then β is an instance of α .

Note. Since β in (ii) is any type-scheme, not just a type, the notion of principal type-scheme here differs slightly from the introduction, where (ii) was only required for types β . However Theorem 1 will show that if X has a type at all, X must have a type-scheme which is principal in the sense of Definition 7, and hence in the sense of the Introduction. If the number of basic types is infinite, I think the two senses are equivalent.

DEFINITION 8. *Principal deductions.* Given an ob X , a deduction \mathcal{D} of a type-scheme of X from a set \mathcal{A} of statements is said to be a *principal deduction* for X iff all other deductions \mathcal{D}' of type-schemes of X are instances of \mathcal{D} (including deductions \mathcal{D}' from other sets \mathcal{A}' of statements). The existence of a principal deduction of αX from a set \mathcal{A} of statements is denoted by

$$\mathcal{A} \vdash_p \alpha X.$$

If X has a principal deduction

$$\{\alpha_1 x_1, \dots, \alpha_n x_n\} \vdash_p \alpha X$$

(where x_1, \dots, x_n are the atoms in X which do not occur in the axioms), then α must be a p.t.s. of X ; because any deduction

$$\{\beta_1 x_1, \dots, \beta_n x_n\} \vdash \beta X$$

must be an instance of the principal deduction, and so $\beta_1, \dots, \beta_n, \beta$ must be instances of $\alpha_1, \dots, \alpha_n, \alpha$, respectively. (In fact, the sequence $\langle \beta_1, \dots, \beta_n, \beta \rangle$ must be an instance of $\langle \alpha_1, \dots, \alpha_n, \alpha \rangle$.)

Notice that for any deduction \mathcal{D} , the type-schemes in its statements can be laid out as a sequence, by starting at the top of the left-hand branch, working down it to just before it joins the next branch, and then going to the top of this next branch, and so on. For instance the sequence for the deduction in Example 1 is \langle type-scheme for **S**, first type-scheme for **K**, type-scheme for **(SK)**, second type-scheme for **K**, type-scheme for **(SKK)** \rangle . Then a deduction is an instance of \mathcal{D} if and only if its sequence is an instance of the sequence for \mathcal{D} (by the same substitution).

2. Lemmas on substitution. These lemmas are stated for substitution into type-schemes, but they are also true for substitution into deductions, and into sequences of type-schemes.

LEMMA 1. *Any substitution*

$$\left[\frac{\beta_1, \dots, \beta_n}{a_1, \dots, a_n} \right] \alpha$$

can be performed by a series of substitutions in which $n=1$, and any series of such "single" substitutions can be performed by one substitution in which $n \geq 1$.

(See notes 7.4 and 7.6, p. 264 of [3].)

COROLLARY. *Two or more successive substitutions can be performed by one substitution; hence an instance of an instance of a type-scheme α is an instance of α .*

LEMMA 2. *Any substitution for variables a_1, \dots, a_n in a type-scheme α can be expressed as a substitution for all the variables in α (by substituting themselves for the variables distinct from a_1, \dots, a_n).*

DEFINITION 9. A *trivial variant* of a type-scheme α is any type-scheme with the form

$$\left[\frac{b_1, \dots, b_n}{a_1, \dots, a_n} \right] \alpha$$

where b_1, \dots, b_n are distinct, and a_1, \dots, a_n include all the variables in α . Trivial variants of deductions, and of sequences of type-schemes, are defined similarly.

LEMMA 3. (a) *The relation of being a trivial variant is reflexive, transitive and symmetric.*

(b) *A type-scheme β is a trivial variant of a type-scheme α if and only if α and β are instances of one another.*

LEMMA 4. (a) *If α is a p.t.s. of X , then any trivial variant of α is a p.t.s. of X , and any p.t.s. of X is a trivial variant of α .*

(b) *Part (a) is also true for principal deductions.*

DEFINITION 10. *Highest common instance.* If α and β are type-schemes, and γ is an instance of both α and β , such that any other instance of both α and β must be an instance of γ , then γ will be called a *highest common instance* (h.c.i.) of α, β . Highest common instances of pairs of sequences of type-schemes are defined similarly.

Notice that if trivial variants are ignored, the h.c.i. of α, β is unique. Also, if α is $\mathbf{F}\alpha_1\alpha_2$ and β is $\mathbf{F}\beta_1\beta_2$, then α, β have a common instance if and only if the sequences $\langle \alpha_1, \alpha_2 \rangle$ and $\langle \beta_1, \beta_2 \rangle$ have a common instance, and if these two sequences have a h.c.i. $\langle \gamma_1, \gamma_2 \rangle$, then $\mathbf{F}\gamma_1\gamma_2$ will be an h.c.i. of α, β .

LEMMA 5 (J. A. ROBINSON). (a) *If two type-schemes α and β have an instance in common, then they have a h.c.i., γ .*

(b) *Furthermore, given any variables a_1, \dots, a_k , γ can be chosen so as to contain none of a_1, \dots, a_k .*

(c) *Parts (a) and (b) are true also for any pair $\langle \alpha_1, \dots, \alpha_p \rangle, \langle \beta_1, \dots, \beta_p \rangle$ of sequences of type-schemes.*

Proof. Part (b) follows from (a) because if γ were an h.c.i. of α, β , any trivial variant of γ would also be an h.c.i. of α, β .

To prove (a), we may assume that α and β have no variables in common, since otherwise we could find trivial variants α' and β' with this property and use them instead of α and β . (By Lemma 3, they would have the same common instances as α and β .) Let a_1, \dots, a_m be all the variables in α , and b_1, \dots, b_n be those in β . Then for any $\gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_n$, since b_1, \dots, b_n do not occur in α , we have

$$\left[\frac{\gamma_1, \dots, \gamma_m}{a_1, \dots, a_m} \right] \alpha \equiv \left[\frac{\gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_n}{a_1, \dots, a_m, b_1, \dots, b_n} \right] \alpha.$$

Similarly,

$$\left[\frac{\delta_1, \dots, \delta_n}{b_1, \dots, b_n} \right] \beta \equiv \left[\frac{\gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_n}{a_1, \dots, a_m, b_1, \dots, b_n} \right] \beta;$$

thus we can look at any common instance of α, β as having been obtained by performing the same substitution on α as on β . Now J. A. Robinson in [4, p. 32] (Unification Theorem) has given an algorithm which allows us to decide whether in the above circumstances, α and β have an instance in common, and if they do, to construct a h.c.i. for them. This proves (a).

To prove (c), notice first that if two sequences $\langle \alpha_1, \dots, \alpha_p \rangle$ and $\langle \beta_1, \dots, \beta_q \rangle$ have an instance in common, then they must have the same number of terms. Let θ be a basic type, and define

$$\begin{aligned} \alpha &\equiv \mathbf{F}\alpha_1(\mathbf{F}\alpha_2(\dots(\mathbf{F}\alpha_p\theta)\dots)) \\ \beta &\equiv \mathbf{F}\beta_1(\mathbf{F}\beta_2(\dots(\mathbf{F}\beta_p\theta)\dots)). \end{aligned}$$

Then α and β have a common instance if and only if the given sequences have a common instance (since nothing can be substituted for θ), and if the h.c.i. of α and β is

$$\mathbf{F}_{\gamma_1}(\mathbf{F}_{\gamma_2}(\dots(\mathbf{F}_{\gamma_p}\theta)\dots)),$$

then the h.c.i. of the given sequences will be

$$\langle \gamma_1, \dots, \gamma_p \rangle.$$

LEMMA 6. *Given a sequence $\langle \phi_1, \dots, \phi_m, \psi_1, \dots, \psi_n \rangle$ of type-schemes, let a_1, \dots, a_p be all the variables in $\langle \phi_1, \dots, \phi_m \rangle$ and b_1, \dots, b_q be the variables (if any) that are in $\langle \psi_1, \dots, \psi_n \rangle$ but not in $\langle \phi_1, \dots, \phi_m \rangle$; let “*” denote the effect of substituting for a_1, \dots, a_p certain type-schemes containing none of b_1, \dots, b_q . Then if $\langle \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \rangle$ is an instance of the given sequence, and $\langle \lambda_1, \dots, \lambda_m \rangle$ is an instance of $\langle \phi_1^*, \dots, \phi_m^* \rangle$, the whole sequence $\langle \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \rangle$ will be an instance of $\langle \phi_1^*, \dots, \phi_m^*, \psi_1^*, \dots, \psi_n^* \rangle$.*

Proof. For ease of reading, suppose $m = n = 1$. Let $\alpha_1, \dots, \alpha_p$ be the type-schemes substituted for a_1, \dots, a_p respectively by the substitution “*”; let e_1, \dots, e_r be all the variables in these type-schemes. Now for some $\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q$,

$$\langle \lambda_1, \mu_1 \rangle \equiv \left[\frac{\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q}{a_1, \dots, a_p, b_1, \dots, b_q} \right] \langle \phi_1, \psi_1 \rangle.$$

But also, for some $\varepsilon_1, \dots, \varepsilon_r$,

$$\begin{aligned} \lambda_1 &\equiv \left[\frac{\varepsilon_1, \dots, \varepsilon_r}{e_1, \dots, e_r} \right] \phi_1^* \\ &\equiv \left[\frac{\varepsilon_1, \dots, \varepsilon_r}{e_1, \dots, e_r} \right] \left[\frac{\alpha_1, \dots, \alpha_p}{a_1, \dots, a_p} \right] \phi_1 \\ &\equiv \left[\frac{\alpha'_1, \dots, \alpha'_p}{a_1, \dots, a_p} \right] \phi_1, \end{aligned}$$

where for each i , α'_i is the result of substituting $\varepsilon_1, \dots, \varepsilon_r$ for e_1, \dots, e_r respectively in α_i . Therefore by the previous expression for $\langle \lambda_1, \mu_1 \rangle$,

$$\gamma_i \equiv \alpha'_i.$$

Hence

$$\begin{aligned} \langle \lambda_1, \mu_1 \rangle &\equiv \left[\frac{\alpha'_1, \dots, \alpha'_p, \delta_1, \dots, \delta_q}{a_1, \dots, a_p, b_1, \dots, b_q} \right] \langle \phi_1, \psi_1 \rangle \\ &\equiv \left[\frac{\varepsilon_1, \dots, \varepsilon_r, \delta_1, \dots, \delta_q}{e_1, \dots, e_r, b_1, \dots, b_q} \right] \left[\frac{\alpha_1, \dots, \alpha_p, b_1, \dots, b_q}{a_1, \dots, a_p, b_1, \dots, b_q} \right] \langle \phi_1, \psi_1 \rangle \end{aligned}$$

since $e_1, \dots, e_r, b_1, \dots, b_q$ are distinct, and $\alpha_1, \dots, \alpha_p$ do not contain b_1, \dots, b_q . Thus $\langle \lambda_1, \mu_1 \rangle$ is an instance of $\langle \phi_1^*, \psi_1^* \rangle$.

3. **Existence of principal type-schemes.** In future, whenever a list (e.g. “ x_1, \dots, x_n ”) of ob-atoms is mentioned, the atoms will be assumed to be distinct.

THEOREM 1. *Let X be an ob, and x_1, \dots, x_t ($t \geq 0$) be the ob-atoms in X that are not given types by the axiom-schemes; if a type-scheme for X can be deduced from some assumptions, then there exists a principal deduction \mathcal{D} for X (and hence X has a p.t.s.).*

Proof. We use induction on the construction of X .

If X is an atom occurring in an axiom-scheme αX , choose \mathcal{D} to be the one-step deduction

$$\alpha X.$$

This is principal because every other deduction must have the form βX , where β is an instance of α .

If X is an atom not given a type by an axiom-scheme, choose \mathcal{D} to be the one-step deduction

$$aX,$$

where a is any type-variable. This is obviously principal. Thus the p.t.s. of X is a single variable in this case.

Now suppose X is (UV) ; let u_1, \dots, u_p be those of x_1, \dots, x_t which occur in U but not in V , w_1, \dots, w_r be those which occur in both U and V , v_1, \dots, v_q be those which occur in V but not in U . Let the given deduction of a type-scheme for X have the form

$$(1) \quad \{\lambda_1 u_1, \dots, \lambda_p u_p, \mu_1 v_1, \dots, \mu_q v_q, \nu_1 w_1, \dots, \nu_r w_r\} \vdash \beta X.$$

This deduction must be the result of applying Rule (F) to two deductions with the forms

$$(2) \quad \{\lambda_1 u_1, \dots, \lambda_p u_p, \nu_1 w_1, \dots, \nu_r w_r\} \vdash \mathbf{F}\alpha\beta U,$$

$$(3) \quad \{\mu_1 v_1, \dots, \mu_q v_q, \nu_1 w_1, \dots, \nu_r w_r\} \vdash \alpha V.$$

By the induction hypothesis, U and V have principal deductions \mathcal{D}_1 and \mathcal{D}_2 ; suppose these have the forms

$$(4) \quad \{\theta_1 u_1, \dots, \theta_p u_p, \psi_1 w_1, \dots, \psi_r w_r\} \vdash_p \pi U,$$

$$(5) \quad \{\phi_1 v_1, \dots, \phi_q v_q, \chi_1 w_1, \dots, \chi_r w_r\} \vdash_p \zeta V.$$

Let a_1, \dots, a_k be the type-variables in \mathcal{D}_1 that do not occur in $\langle \psi_1, \dots, \psi_r, \pi \rangle$,

b_1, \dots, b_l be all the type-variables in $\langle \psi_1, \dots, \psi_r, \pi \rangle$,

c_1, \dots, c_m be all the type-variables in $\langle \chi_1, \dots, \chi_r, \zeta \rangle$,

d_1, \dots, d_n be the type-variables in \mathcal{D}_2 that are not in $\langle \chi_1, \dots, \chi_r, \zeta \rangle$. Let g be a new type-variable, distinct from all those above.

The principal deduction for (UV) is constructed as follows. First choose \mathcal{D}_1 and \mathcal{D}_2 so that all the above variables are distinct (this is possible by Lemma 4(b)).

Then find an h.c.i. of the two sequences

$$\langle \psi_1, \dots, \psi_r, \pi \rangle, \quad \langle \chi_1, \dots, \chi_r, (\mathbf{F}\zeta g) \rangle$$

which contains none of $a_1, \dots, a_k, d_1, \dots, d_n$ (this is possible by Lemma 5(c), since the two sequences have a common instance $\langle \nu_1, \dots, \nu_r, (\mathbf{F}\alpha\beta) \rangle$ by (2) and (3) above). Suppose this h.c.i. is

$$(6) \quad \langle \omega_1, \dots, \omega_r, (\mathbf{F}\sigma\tau) \rangle, \equiv \langle \psi_1^*, \dots, \psi_r^*, \pi^* \rangle, \equiv \langle \chi_1, \dots, \chi_r^\circ, (\mathbf{F}\zeta^\circ\tau) \rangle$$

where “*” denotes the substitution of certain type-schemes for b_1, \dots, b_1 only, and “ $^\circ$ ” for c_1, \dots, c_m only. Then construct the two deductions \mathcal{D}_1^* and \mathcal{D}_2° , which give

$$(7) \quad \{ \theta_p^* u_1, \dots, \theta_p^* u_p, \psi_1^* w_1, \dots, \psi_r^* w_r \} \vdash \mathbf{F}\sigma\tau U,$$

$$(8) \quad \{ \phi_1^\circ v_1, \dots, \phi_q^\circ v_q, \chi_1^\circ w_1, \dots, \chi_r^\circ w_r \} \vdash \zeta^\circ V.$$

Since $\sigma \equiv \zeta^\circ$ and $\psi_i^* \equiv \chi_i^\circ$ for each i , these two deductions can be combined by Rule (F) to form one deduction, giving

$$(9) \quad \{ \theta_p^* u_1, \dots, \theta_p^* u_p, \psi_1^* w_1, \dots, \psi_r^* w_r, \phi_1^\circ v_1, \dots, \phi_q^\circ v_q \} \vdash \tau(UV).$$

We shall see that this is a principal deduction for (UV) ; to do this, we must prove that any deduction with the form (1) is an instance of (9)⁽⁶⁾. For this it is enough to show that the deduction (2) is an instance of (7), and (3) is an instance of (8), both by the same substitution. Notice that since (4) and (5) are principal, (2) and (3) are instances of (4) and (5) respectively.

Now let $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ be the sequences for the deductions (2), (3), (4), (5) respectively (see the end of §1). To show that (2), (3) are instances of (7), (8) both by the same substitution, it is enough to show that the sequence $\langle \mathcal{S}_2, \mathcal{S}_3 \rangle$ (formed by putting the members of \mathcal{S}_3 after the members of \mathcal{S}_2) is an instance of the sequence $\langle \mathcal{S}_4^*, \mathcal{S}_5^\circ \rangle$, since \mathcal{S}_4^* is the sequence for (7), and \mathcal{S}_5° corresponds to (8). We shall use Lemma 6.

Since no variables are common to (4) and (5), and (2), (3) are instances of (4), (5), the sequence $\langle \mathcal{S}_2, \mathcal{S}_3 \rangle$ is an instance of $\langle \mathcal{S}_4, \mathcal{S}_5 \rangle$. Hence the sequence

$$\langle \mathcal{S}_2, \mathcal{S}_3, \nu_1, \dots, \nu_r, (\mathbf{F}\alpha\beta), \nu_1, \dots, \nu_r, (\mathbf{F}\alpha g) \rangle$$

is an instance of

$$\langle \mathcal{S}_4, \mathcal{S}_5, \psi_1, \dots, \psi_r, \pi, \chi_1, \dots, \chi_r, (\mathbf{F}\zeta g) \rangle$$

because $\nu_1, \dots, \nu_r, (\mathbf{F}\alpha\beta)$ occur in \mathcal{S}_2 at positions corresponding to $\psi_1, \dots, \psi_r, \pi$ in \mathcal{S}_4 , and similarly for the other members. Also, by (6),

⁽⁶⁾ Here the relevant deductions will be called “(1),” “(9),” etc., though strictly speaking (1) and (9) are not deductions but only the hypotheses and conclusions of deductions. Thus for instance, type-variables may occur in the deduction giving (1) but not appear in (1) itself (if they occurred in axioms used in the deduction); in Example 1, b is such a variable.

$$\begin{aligned} \langle \omega_1, \dots, \omega_r, (\mathbf{F}\sigma\tau), \omega_1, \dots, \omega_r, (\mathbf{F}\sigma\tau) \rangle &\equiv \langle \psi_1^*, \dots, \psi_r^*, \pi^*, \chi_1^\circ, \dots, \chi_r^\circ, (\mathbf{F}\zeta^\circ\tau) \rangle \\ &\equiv \langle \psi_1, \dots, \psi_r, \pi, \chi_1, \dots, \chi_r, (\mathbf{F}\zeta g) \rangle^+ \end{aligned}$$

where “+” denotes the result of simultaneously performing substitution * on b_1, \dots, b_1 , and $^\circ$ on c_1, \dots, c_m , and substituting τ for g . By the definition of (6), if there are any type-variables in \mathcal{S}_4 or \mathcal{S}_5 which do not occur in

$$\langle \psi_1, \dots, \psi_r, \pi, \chi_1, \dots, \chi_r, (\mathbf{F}\zeta g) \rangle,$$

they do not occur in the above sequence. Hence by Lemma 6,

$$\langle \mathcal{S}_2, \mathcal{S}_3, \nu_1, \dots, \nu_r, (\mathbf{F}\alpha\beta), \nu_1, \dots, \nu_r, (\mathbf{F}\alpha g) \rangle$$

is an instance of

$$\langle \mathcal{S}_4^+, \mathcal{S}_5^+, \psi_1^+, \dots, \psi_r^+, \pi^+, \chi_1^+, \dots, \chi_r^+, (\mathbf{F}\zeta g)^+ \rangle.$$

Hence $\langle \mathcal{S}_2, \mathcal{S}_3 \rangle$ is an instance of $\langle \mathcal{S}_4^+, \mathcal{S}_5^+ \rangle$; since this is the same as $\langle \mathcal{S}_4^*, \mathcal{S}_5^\circ \rangle$ the proof is complete.

REMARK. In this proof the p.t.s. π of U in (4) may be either compound or a variable. (It cannot be a basic type since it has an instance $\mathbf{F}\alpha\beta$ which is compound.) The variable g was introduced just to treat these two cases simultaneously, and when π is compound, g is not really necessary.

COROLLARY 1.1. *There is an effective way of deciding for each X whether or not X has a type-scheme, and if it has one, for calculating its p.t.s., and its principal deduction. (Assuming that atoms can be effectively distinguished from each other and from compound obs, and that substitutions can be effectively performed, etc.)*

Proof. If X is an atom, then the p.t.s. of X either is given by an axiom-scheme, or else is a single type-variable.

Suppose X is (UV) ; if U or V has no type-schemes, then X has none. If U and V have principal deductions with the forms (4) and (5), then Robinson’s theorem (Lemma 5) gives an effective test of whether the h.c.i. (6) is constructible; if it is, the principal deduction for X is (9) and the p.t.s. is τ therein; if not, X has no type-schemes at all.

COROLLARY 1.2. *For every p.t.s. α of X , there exists a principal deduction of αX .*

Proof. By Theorem 1, X has a principal deduction; let βX be the statement it deduces. By Lemma 4(a), β is a trivial variant of α , and hence suitable changes of the variables in the deduction of βX will form a principal deduction of αX .

The following two corollaries show how to calculate the p.t.s. of (UV) from those of U and V , in the important case that all atoms common to U and V are given types by the axiom-schemes. In this case $r=0$ in the proof of the theorem, and so the principal deduction is constructed by finding an h.c.i. of π and $\mathbf{F}\zeta g$, and combining the corresponding deductions (7) and (8).

COROLLARY 1.3. *Suppose that all the atoms common to U and V are given types by the axiom-schemes; if*

$$(4a) \quad \{\theta_1 u_1, \dots, \theta_p u_p\} \vdash_p \mathbf{F}\xi\eta U$$

and

$$(5a) \quad \{\phi_1 v_1, \dots, \phi_q v_q\} \vdash_p \zeta V$$

and ξ, ζ have an h.c.i. $\xi^* \equiv \zeta^\circ$ whose new variables (i.e. those not in ξ) do not occur in η , then the result η^* , of making exactly the same substitutions in η as changed ξ to ξ^* , is a p.t.s. of (UV) .

Proof. By Lemma 4b we may replace the deduction (5a) by one with no variables in common with (4a). Suppose this deduction gives

$$(5a)' \quad \{\phi'_1 v_1, \dots, \phi'_q v_q\} \vdash_p \zeta' V.$$

The type-scheme ξ^* will be an h.c.i. of ξ and ζ' .

Now $\mathbf{F}\xi^*\eta^*$ is an h.c.i. of $\mathbf{F}\xi\eta$ and $\mathbf{F}\zeta'g$, if g does not occur in ζ' . It is certainly a common instance. Also if $\mathbf{F}\alpha\beta$ is any other common instance, α must be an instance of ξ^* since ξ^* is an h.c.i. of ξ and ζ' ; hence by Lemma 6, $\mathbf{F}\alpha\beta$ is an instance of $\mathbf{F}\xi^*\eta^*$. (Lemma 6 is here applied to the pairs $\langle \xi, \eta \rangle$, $\langle \xi^*, \eta^* \rangle$ and $\langle \alpha, \beta \rangle$.)

To follow the construction in the proof of Theorem 1, with $r=0$ and $\pi \equiv \mathbf{F}\xi\eta$ in the present case, the first step is to find an h.c.i. of $\mathbf{F}\xi\eta$ and $\mathbf{F}\zeta'g$ that does not contain $a_1, \dots, a_k, d_1, \dots, d_n$ (see Theorem 1). If $\mathbf{F}\xi^*\eta^*$ is not suitable, we can find a suitable trivial variant $\mathbf{F}\xi''\eta''$ of it. Then the principal deduction (9) for (UV) is formed by combining certain instances of (4a) and (5a)', and shows that η'' is a p.t.s. of UV . Hence η^* is a p.t.s., since it is a trivial variant of η'' .

COROLLARY 1.4. *If in Corollary 1.3 all the variables in the hypotheses of (4a) occur in $\mathbf{F}\xi\eta$, and all those in the hypotheses of (5a) occur in ζ , then*

$$\{\theta_1^* u_1, \dots, \theta_p^* u_p, \phi_1^\circ v_1, \dots, \phi_q^\circ v_q\} \vdash_p \eta^*(UV).$$

Proof. In the proof of Corollary 1.3, a_1, \dots, a_k are the variables in the deduction for U which do not occur in $\mathbf{F}\xi\eta$; hence by the assumption they do not occur in any of $\theta_1, \dots, \theta_p, \mathbf{F}\xi\eta$. Therefore by Lemma 4b a principal deduction can be found which still gives (4a) but with a_1, \dots, a_k not occurring in $\mathbf{F}\xi^*\eta^*$. Similarly we may assume that d_1, \dots, d_n in (5a)' do not occur in $\mathbf{F}\xi^*\eta^*$. Therefore $\mathbf{F}\xi^*\eta^*$ is a suitable h.c.i. for the proof of Corollary 1.3. To construct the principal deduction (9) for UV , we first perform in (5a)' the substitution which changes ζ' to ζ° ; since all the variables in each ϕ'_i occur in ζ' , this changes ϕ'_i to ϕ_i° , giving a deduction

$$\{\phi_1^\circ v_1, \dots, \phi_q^\circ v_q\} \vdash_p \zeta^\circ V.$$

Next we perform substitution “*” in (4a) and combine the result with the above deduction by Rule (F), completing the corollary.

COROLLARY 1.5. *If in (4) and (5) of Theorem 1, π has the form $\mathbf{F}\xi\eta$ and $\zeta \equiv \xi$, $\psi_i \equiv \chi_i$ for all i (these being the only exceptions to the assumption that (4) and (5) have no variables in common), then a principal deduction for UV is obtained by simply applying Rule (F) to (4) and (5).*

Proof. To construct a principal deduction as in Theorem 1, first change all the variables in (5) that occur in $\zeta, \chi_1, \dots, \chi_r$ to distinct new ones not occurring in (4). This leaves (4) and (5) with no variables in common. Then in (6), take “*” to be the identity substitution and “ \circ ” to be the reverse of the above change of variables, and $\tau \equiv \eta$. Then (7) and (8) are the given (4) and (5); hence the result.

EXAMPLE 2. Let x, y, z be distinct ob-atoms not given types by the axiom-schemes; then

$$\{\mathbf{F}ab\ x, a\ y, a\ z\} \vdash_p b(\mathbf{K}(xy)(xz)).$$

Proof. Let a, b, c, d, e, g_1, g_2 be distinct type-variables. Principal deductions for x and y are

$$c\ x, \quad a\ y.$$

To get a principal deduction for (xy) by the construction in Theorem 1, notice that an h.c.i. of c and $(\mathbf{F}ag_1)$ is just $(\mathbf{F}ag_1)$ itself; hence, replacing c by $(\mathbf{F}ag_1)$ and combining the two deductions by Rule (F) gives a principal deduction

$$(10) \quad \frac{\mathbf{F}ag_1x \quad a\ y}{g_1(xy)}$$

A principal deduction for \mathbf{K} is the one step $\mathbf{F}g_1(\mathbf{F}bg_1)\mathbf{K}$; combining this with (10) by Corollary 1.5 gives the principal deduction

$$(11) \quad \frac{\mathbf{F}g_1(\mathbf{F}bg_1)\mathbf{K} \quad \frac{\mathbf{F}ag_1x \quad a\ y}{g_1(xy)}}{\mathbf{F}bg_1(\mathbf{K}(xy))}$$

Now just as for (xy) , a principal deduction for (xz) is

$$(12) \quad \frac{\mathbf{F}ed\ x \quad e\ z}{d(xz)}$$

To obtain the result, combine (11) and (12). No type-variables are common to both, and the only ob-atom they have in common is x ; hence we must find an h.c.i. of the sequences

$$\langle\langle \mathbf{F}ag_1, (\mathbf{F}bg_1) \rangle\rangle, \quad \langle\langle \mathbf{F}cd, (\mathbf{F}dg_2) \rangle\rangle.$$

All variables in (10) and (11) occur in these sequences, so there is no restriction on the h.c.i. we can use. A suitable one is

$$\langle\langle \mathbf{F}ab, (\mathbf{F}bb) \rangle\rangle,$$

obtained by substituting b for g_1 in the former sequence, and a, b, b for c, d, g_2 in the latter sequence. Making the same substitutions in (11) and (12) and combining by Rule (F) gives the principal deduction

$$\{\mathbf{F}ab\ x, a\ y, a\ z\} \vdash_p b(\mathbf{K}(xy)(xz)).$$

EXAMPLE 3. If $\mathbf{I} \equiv \mathbf{SKK}$, then $\vdash_p \mathbf{F}aal$.

Proof. By the axiom-schemes,

$$\vdash_p \mathbf{F}(\mathbf{F}a(\mathbf{F}bc))(\mathbf{F}(\mathbf{F}ab)(\mathbf{F}ac))\mathbf{S}, \quad \vdash_p \mathbf{F}a(\mathbf{F}ba)\mathbf{K}.$$

By Corollary 1.3, since $\mathbf{F}a(\mathbf{F}ba)$ is an h.c.i. of $\mathbf{F}a(\mathbf{F}bc)$ and $\mathbf{F}a(\mathbf{F}ba)$, substituting a for c in the p.t.s. of \mathbf{S} and applying Rule (F) gives

$$\vdash_p \mathbf{F}(\mathbf{F}ab)(\mathbf{F}aa)(\mathbf{SK}).$$

Now $\mathbf{F}a(\mathbf{F}ba)$ is an h.c.i. of $\mathbf{F}ab$ and $\mathbf{F}a(\mathbf{F}ba)$, so by Corollary 1.3 again (substituting $\mathbf{F}ba$ for b in the p.t.s. of \mathbf{SK}),

$$\vdash_p \mathbf{F}aa(\mathbf{SKK}).$$

But we can say more than this: in the first application of Corollary 1.3 above, there are no $a_1, \dots, a_k, d_1, \dots, d_n$ to worry about in the proof of the corollary, and similarly for the second application (since every variable in the deduction for \mathbf{SK} occurs in its conclusion), so the above proof actually gives us a principal deduction for \mathbf{SKK} . This deduction is the same as in Example 1.

EXAMPLE 4. The ob \mathbf{SII} has no type-schemes at all.

Proof. By Example 3,

$$\vdash_p \mathbf{F}(\mathbf{F}a(\mathbf{F}bc))((\mathbf{F}ab)(\mathbf{F}ac))\mathbf{S}, \quad \vdash_p \mathbf{F}aal.$$

Now $\mathbf{F}(\mathbf{F}bc)(\mathbf{F}bc)$ is an h.c.i. of $\mathbf{F}a(\mathbf{F}bc)$ and $\mathbf{F}aa$, so by Corollary 1.3 and substituting $\mathbf{F}bc$ for a ,

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}bc)b)(\mathbf{F}(\mathbf{F}bc)c)(\mathbf{SI}).$$

If \mathbf{SII} had any type-schemes, the construction in Theorem 1 would give us a p.t.s. But this construction involves finding an h.c.i. of

$$\mathbf{F}(\mathbf{F}(\mathbf{F}bc)b)(\mathbf{F}(\mathbf{F}bc)c), \quad \mathbf{F}(\mathbf{F}aa)g,$$

and the existence of such an h.c.i. would imply that

$$\mathbf{F}(\mathbf{F}bc)b, \quad \mathbf{F}aa$$

have instances in common, which is impossible. Thus the construction fails, so \mathbf{SII} has no type-schemes.

4. **The principal type-scheme of $[x].M$.** The main result of this section is a lemma for use in the proof of Theorem 3, §6. From now on the ob-atoms will be assumed to include \mathbf{S} and \mathbf{K} , together with some *ob-variables*, which are not given

types by the axiom-schemes. (These variables can be adjoined as extra ob-atoms if they are not already present.) Letters “ u ”, “ v ”, “ w ”, “ x ”, “ y ”, “ z ” will denote these variables, and for ease of reading it will be assumed that all the ob-atoms which do not occur in the axiom-schemes are variables.

DEFINITION 11. *Reduction and equality.* An ob X is said to *reduce to* Y (“ $X \geq Y$ ”) iff either $Y \equiv X$ or else Y is the result of applying to X a series of replacements with the forms (for any U, V, W):

S UVW may be replaced by $UW(VW)$,

K UV may be replaced by U .

The properties and purpose of reduction are explained in [1]. The phrase “ X is equal to Y ” (“ $X = Y$ ”) is defined to mean that Y is the result of applying to X a series of replacements as above, and reversed replacements.

It is proved in [1, §9C2, Theorem 2] that

$$(13) \quad \text{if } X \geq Y \text{ and } \{\alpha_1 x_1, \dots, \alpha_n x_n\} \vdash \beta X, \text{ then } \{\alpha_1 x_1, \dots, \alpha_n x_n\} \vdash \beta Y.$$

DEFINITION 12. $[x].M$. For each ob-variable x and each ob M , an ob called “ $[x].M$ ” can be defined as follows, by induction on the construction of M .

(i) $[x].x \equiv \mathbf{I}$, that is, **SKK**;

(ii) $[x].M \equiv \mathbf{KM}$ if M does not contain x ;

(iii) $[x].(UV) \equiv ((\mathbf{S}[x].U)[x].V)$ if (UV) contains x .

This definition has the property that for all x, M, N ,

$$(14) \quad ([x].M)N \geq [N/x]M,$$

where $[N/x]M$ is just the result of substituting N for every occurrence of x in M . Thus $[x].M$ represents the function of x defined by M ; see [1, §6A] for details. Since $[x].M$ represents a function of x , we should expect to be able to prove that if M has type η , deduced by giving x a type ξ , then $[x].M$ has type $\mathbf{F}\xi\eta$. More precisely, letting y_1, \dots, y_n be the other ob-variables in M ,

$$(15) \quad \text{if for some types } \alpha_1, \dots, \alpha_n, \xi, \eta, \{\alpha_1 y_1, \dots, \alpha_n y_n, \xi x\} \vdash \eta M, \\ \text{then } \{\alpha_1 y_1, \dots, \alpha_n y_n\} \vdash \mathbf{F}\xi\eta([x].M).$$

This is actually proved in [1, §9D, Corollary 1.1]; the theorem below is its analogue for principal deductions.

THEOREM 2. (a) *Suppose that x, y_1, \dots, y_n are the only ob-variables in M . If*

$$\{\alpha_1 y_1, \dots, \alpha_n y_n, \xi x\} \vdash_p \eta M,$$

then

$$\{\alpha_1 y_1, \dots, \alpha_n y_n\} \vdash_p \mathbf{F}\xi\eta([x].M).$$

(b) *Suppose that M contains only y_1, \dots, y_n and not x ; if*

$$\{\alpha_1 y_1, \dots, \alpha_n y_n\} \vdash_p \eta M,$$

then

$$\{\alpha_1 y_1, \dots, \alpha_n y_n\} \vdash_p \mathbf{F}e\eta([x].M)$$

where e is any type-variable not occurring in η .

Proof. For part (b): $[x].M$ is \mathbf{KM} , and

$$\vdash_p \mathbf{Fa}(\mathbf{Fea})K.$$

Therefore by the given deduction and Corollary 1.3, since η is an h.c.i. of η and a ,

$$\{\alpha_1\gamma_1, \dots, \alpha_n\gamma_n\} \vdash_p \mathbf{Fe}\eta(\mathbf{KM}).$$

For part (a) we use Curry's proof (see an earlier comment). Firstly, since M has a type-scheme it must have a type, since all the type-variables in a deduction can be replaced by basic types. Therefore by (15), $[x].M$ has a type. Every type of $[x].M$ must be compound; this is true if M is x , since then $[x].M$ is \mathbf{I} ; when M has the form UV , then

$$[x].M \equiv (\mathbf{S}[x].U)[x].V$$

and by the axiom-scheme for \mathbf{S} , the p.t.s. of this ob (if it has one) must be an instance of (\mathbf{Fac}) , which is compound. Hence the p.t.s. of $[x].M$ is compound, so the principal deduction for $[x].M$ must have the form

$$(16) \quad \{\beta_1\gamma_1, \dots, \beta_n\gamma_n\} \vdash_p \mathbf{F}\gamma\delta([x].M).$$

By (15) and the given deduction for M we have a deduction

$$(17) \quad \{\alpha_1\gamma_1, \dots, \alpha_n\gamma_n\} \vdash \mathbf{F}\xi\eta([x].M).$$

This must be an instance of (16), since (16) is principal; hence

$$(18) \quad \langle \alpha_1, \dots, \alpha_n, (\mathbf{F}\xi\eta) \rangle \text{ is an instance of } \langle \beta_1, \dots, \beta_n, (\mathbf{F}\gamma\delta) \rangle.$$

Now by (16) and Rule (F),

$$\{\beta_1\gamma_1, \dots, \beta_n\gamma_n, \gamma x\} \vdash \delta([x].M)x,$$

and so by (14) and (13),

$$\{\beta_1\gamma_1, \dots, \beta_n\gamma_n, \gamma x\} \vdash \delta M.$$

This deduction must be an instance of the given principal deduction for M , and hence $\langle \beta_1, \dots, \beta_n, (\mathbf{F}\gamma\delta) \rangle$ is an instance of $\langle \alpha_1, \dots, \alpha_n, (\mathbf{F}\xi\eta) \rangle$. Therefore by (18) these two sequences are trivial variants of one another. Now we construct a principal deduction with the same hypotheses and conclusion as (17). To do this, replace all type-variables in the deduction (16) which do not occur in

$$\langle \beta_1, \dots, \beta_n, (\mathbf{F}\gamma\delta) \rangle$$

by distinct new variables not occurring in $\langle \alpha_1, \dots, \alpha_n, (\mathbf{F}\xi\eta) \rangle$, and then make the substitutions which change $\langle \beta_1, \dots, \beta_n, (\mathbf{F}\gamma\delta) \rangle$ to $\langle \alpha_1, \dots, \alpha_n, (\mathbf{F}\xi\eta) \rangle$. The result is a deduction giving (17), and it is principal because it is a trivial variant of (16).

COROLLARY 2.1. *If the only ob-variables in M are x_1, \dots, x_n ($n \geq 1$), and*

$$\{\alpha_1x_1, \dots, \alpha_nx_n\} \vdash_p \beta M,$$

then

$$\vdash_p (\mathbf{F}_n \alpha_1 \cdots \alpha_n \beta)([x_1, \dots, x_n].M),$$

where

$$(\mathbf{F}_n \alpha_1 \cdots \alpha_n \beta) \text{ is } \mathbf{F}_{\alpha_1}(\mathbf{F}_{\alpha_2}(\cdots(\mathbf{F}_{\alpha_n} \beta) \cdots))$$

or just β when $n=0$, and

$$[x_1, \dots, x_n].M \equiv [x_1].([x_2].(\cdots([x_n].M) \cdots))$$

or just M when $n=0$.

EXAMPLE 5. Define

$$A \equiv [x, y, z].\mathbf{K}(xy)(xz):$$

then by Example 2 and Corollary 2.1,

$$\vdash_p \mathbf{F}(\mathbf{F}ab)(\mathbf{F}a(\mathbf{F}ab))A.$$

REMARK. H. Curry has pointed out that if the definition of $[x].M$ included, as it usually does, the clause

(iv) $[x].(Ux) \equiv U$ when U does not contain x

(and clause (iii) were only applied when (iv) was not applicable), then Theorem 2 would be false. For example the deduction

$$\frac{\mathbf{F}ab \quad y \quad ax}{b(yx)}$$

is a principal one for (yx) , and by (iv),

$$[x].(yx) \equiv y,$$

so Theorem 2 would say that

$$\mathbf{F}ab \vdash_p \mathbf{F}ab \quad y,$$

which is false because the one-step deduction

$$\mathbf{F}ab \quad y$$

is not a principal one for y . The proof of Theorem 2 breaks down when (iv) is used because then the p.t.s. of $[x].M$ may not be compound; but this does not mean that Theorem 2 is useless; see the proof of Theorem 3 later.

Incidentally, Theorem 2 can be proved by induction on the construction of M without using (14) and (15); though this proof is longer than the above one, it does give an alternative proof of (15), because (15) follows from Theorems 1 and 2.

5. An alternative approach to typed combinators. Now besides the formal system set up in Definitions 1 to 4, there is another way of introducing type-restrictions into combinatory logic. In Definition 1, instead of postulating two basic combinators \mathbf{S} and \mathbf{K} , we could postulate an infinite set of basic combinators.

For each ordered triple α, β, γ of types there would be a distinct basic combinator $\mathbf{S}_{\alpha\beta\gamma}$, and for each pair α, β there would be a distinct basic $\mathbf{K}_{\alpha\beta}$; the axiom-schemes for \mathbf{S} and \mathbf{K} in Definition 4 would be replaced by the infinite set of axioms

$$\mathbf{F}(\mathbf{F}\alpha(\mathbf{F}\beta\gamma))(\mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}\alpha\gamma))\mathbf{S}_{\alpha\beta\gamma}$$

$$\mathbf{F}\alpha(\mathbf{F}\beta\alpha)\mathbf{K}_{\alpha\beta}.$$

The type-variables in Definition 2 would not be used at all.

It can be seen that each combinator in this alternative system would have just one type (if it had a type at all), in contrast to the infinite number of types given to each combinator in the former system. Hence it is easier to give a set-theoretic interpretation to the combinators in the alternative system than in the original system; for instance $\mathbf{K}_{\alpha\beta}$ is interpreted as the function f such that (in the usual function notation)

$$(f(x))(y) = x$$

for each member x of the set α , and each member y of β . This f changes its argument x into a constant-valued function. But in the former system, the basic combinator \mathbf{K} did not represent a particular function f , but rather the abstract notion of forming a constant function from any object; this is much harder to interpret set-theoretically. For instance \mathbf{K} can be applied to itself; [more precisely, $(\mathbf{K}\mathbf{K})$ is a combinator and it even has a p.t.s. $\mathbf{F}\alpha(\mathbf{F}\beta(\mathbf{F}cb))$] but the set-theoretic meaning of applying \mathbf{K} to itself is not so clear. Actually the combinators perform their tasks without needing any such interpretation (compare the parentheses and punctuation symbols in the usual forms of predicate calculus), but if a set-theoretic interpretation is wanted, then it seems more convenient to use the system based on an infinity of combinators $\mathbf{S}_{\alpha\beta\gamma}$, $\mathbf{K}_{\alpha\beta}$ (see [5], for example).

However, most of the fundamental syntactical properties of the combinators in this alternative system are independent of the types of the particular $\mathbf{S}_{\alpha\beta\gamma}$ and $\mathbf{K}_{\alpha\beta}$ involved. So for discussing these fundamental properties the original system is more natural.

In the alternative system, reduction and equality are defined as in Definition 11, except that $(\mathbf{K}_{\alpha\beta}UV)$ may be replaced by U only when

$$\vdash \alpha U \quad \text{and} \quad \vdash \beta V,$$

and $(\mathbf{S}_{\alpha\beta\gamma}UVW)$ may be replaced by $UW(VW)$ only when

$$\vdash \mathbf{F}\alpha(\mathbf{F}\beta\gamma)U \quad \text{and} \quad \vdash \mathbf{F}\alpha\beta V \quad \text{and} \quad \vdash \alpha W.$$

These conditions ensure that an ob with a type is replaced by another with the same type; hence if $X \geq Y$ or $X = Y$, then X and Y both have the same type (and if either has a type, the other does).

Now it looks as if it might be possible to find a subsystem of the original system based on \mathbf{S} and \mathbf{K} to correspond to the infinite-based system, by using the fact

that each ob with a type has a unique p.t.s. (neglecting trivial variants) and somehow making principal type-schemes in the original system do the work of types in the infinite-based system. An ob in the finite-based system whose p.t.s. is a type α can have only one type, since all its types are instances of α , which contains no variables. Therefore we could try to set up a correspondence between infinite-based obs, and finite-based obs whose principal type-schemes are types. One way of doing this is as follows (letters “ X ”, “ Y ” will denote finite-based obs here, and “ M ”, “ N ”, “ P ” infinite-based obs).

Firstly, suppose we could prove

$$(19) \quad \text{If } \vdash_{\infty} \alpha X \text{ in the system of } \S 1, \text{ then there is} \\ \text{an ob } X_{\alpha} \text{ such that } \vdash_p \alpha X_{\alpha} \text{ and } X_{\alpha} \geq X.$$

Then applying (19) to \mathbf{K} and \mathbf{S} would give us, for each triple α, β, γ of types, two obs $\mathbf{K}_{\alpha\beta}^*$ and $\mathbf{S}_{\alpha\beta\gamma}^*$ with the principal type-schemes, respectively,

$$\mathbf{F}_{\alpha}(\mathbf{F}\beta\alpha), \quad \mathbf{F}(\mathbf{F}\alpha(\mathbf{F}\beta\gamma))(\mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}\alpha\gamma)).$$

These two type-schemes contain no variables, so they are the only types that $\mathbf{K}_{\alpha\beta}^*$ and $\mathbf{S}_{\alpha\beta\gamma}^*$ could have. Hence to each combinator P (with a type δ) in the infinite-based system, would correspond a finite-based ob P^T whose only type was δ ; P^T would be defined thus:

$$(\mathbf{S}_{\alpha\beta\gamma})^T \equiv \mathbf{S}_{\alpha\beta\gamma}^*, \quad (\mathbf{K}_{\alpha\beta})^T \equiv \mathbf{K}_{\alpha\beta}^*, \quad (MN)^T \equiv (M^T N^T).$$

To complete the embedding of the infinite-based system in the finite-based one, we would only need to show that

(20) $M \geq N$ in the infinite-based system iff $M^T \geq N^T$ in the finite-based system. Half of this equivalence follows from (19); in fact, to show that $M \geq N$ implies $M^T \geq N^T$, it is enough to show that for all M, N, P ,

$$\mathbf{S}_{\alpha\beta\gamma}^* M^T N^T P^T \geq M^T P^T (N^T P^T), \quad \mathbf{K}_{\alpha\beta}^* M^T N^T \geq M^T$$

in the finite-based system. These are true, because

$$\begin{aligned} \mathbf{K}_{\alpha\beta}^* M^T N^T &\geq \mathbf{K} M^T N^T \text{ by (19)} \\ &\geq M^T \text{ by Definition 11,} \end{aligned}$$

and similarly for $\mathbf{S}_{\alpha\beta\gamma}^*$.

The embedding can also be seen to be one-to-one, using the actual structure of $\mathbf{K}_{\alpha\beta}^*$ and $\mathbf{S}_{\alpha\beta\gamma}^*$ given by the proof of (19) in the next section.

However I do not know if the converse half of (20) is true⁽⁷⁾.

(7) The main use of the alternative system (with each atom having a unique type) is to represent certain sets S of functionals (see [5] for example). We represent each ϕ in S with type α by an ob $[\phi]$ with the same type, such that $[\phi][\psi] = [\phi(\psi)]$ if ϕ has type $F\beta\gamma$ and ψ is in S with type β and $\phi(\psi)$ is in S . One purpose of the above embedding (suggested by Curry) is to show that every ϕ which can be represented as above can also be represented by a finite-based ob $[\phi]'$ such that $[\phi]'\psi' = [\phi(\psi)]'$. For this result we do not need the converse half of (20); we simply define $[\phi]'$ to be $[\phi]^T$.

The proof of (19) will be given in the next section (Corollary 3.1). Although it is only used in the above argument in the case that α is a type, it seems easiest to prove it first for the case that α contains only variables, and then extend it to the other case. This extension assumes that each basic type is the p.t.s. of at least one ob (not necessarily an ob-atom).

6. Every type-scheme is a p.t.s. The main purpose of this section is to verify (19), which seems interesting in its own right, besides its application in the preceding section.

The obs of this section are those of §1–4, not the alternative system in §5. Type-variables denoted by distinct letters will be assumed to be distinct, and calling a variable “new” will mean that it does not occur in any type-scheme mentioned earlier in the section.

LEMMA 7. *Given any distinct type-variables b, c, d, e , define*

$$\begin{aligned} \xi_1 &\equiv \mathbf{F}(\mathbf{F}de)c & \eta_1 &\equiv \mathbf{F}dc \\ \xi_2 &\equiv \mathbf{F}(\mathbf{F}de)c & \eta_2 &\equiv \mathbf{F}ec \\ \xi_3 &\equiv \mathbf{F}b(\mathbf{F}de) & \eta_3 &\equiv \mathbf{F}bd \\ \xi_4 &\equiv \mathbf{F}b(\mathbf{F}de) & \eta_4 &\equiv \mathbf{F}be. \end{aligned}$$

Then there exist combinators B_1, \dots, B_4 such that for $i = 1, \dots, 4$,

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_i\xi_i)(\mathbf{F}(\mathbf{F}\eta_i\eta_i)(\mathbf{F}\xi_i\xi_i))B_i.$$

Also there is a combinator C such that

$$\vdash_p \mathbf{F}(\mathbf{F}bb)(\mathbf{F}(\mathbf{F}cc)(\mathbf{F}(\mathbf{F}bc)(\mathbf{F}bc)))C.$$

Proof. Tedious; see the end of the section.

THEOREM 3. *If X is an ob whose atoms are all given types by the axioms, and*

$$\vdash_\alpha X$$

for some α containing only variables, then there exists an ob X_α such that

$$X_\alpha \geq X \text{ and } \vdash_p \alpha X_\alpha;$$

if X is a combinator, then so is X_α .

Proof. First suppose that for each α containing only variables there exists a combinator V_α such that

$$\vdash_p \mathbf{F}\alpha V_\alpha.$$

Then using the ob A from Example 5, §4, define

$$X_\alpha \equiv \mathbf{S}(A\mathbf{I})V_\alpha X.$$

Then

$$\begin{aligned} X_\alpha &\geq A\mathbf{I}X(V_\alpha X) \\ &\geq \mathbf{K}(\mathbf{I}X)(\mathbf{I}(V_\alpha X)) \subset \text{by (14) and the def. of } A \\ &\geq X. \end{aligned}$$

Also we have by Examples 5 and 3,

$$\vdash_p \mathbf{F}(\mathbf{F}ab)(\mathbf{F}a(\mathbf{F}ab))A \quad \vdash_p \mathbf{F}aal.$$

Therefore by Corollary 1.3, substituting a for b ,

$$\vdash_p \mathbf{F}a(\mathbf{F}aa)(A).$$

Now by the axiom-schemes,

$$\vdash_p \mathbf{F}(\mathbf{F}a(\mathbf{F}bc))(\mathbf{F}(\mathbf{F}ab)(\mathbf{F}ac))\mathbf{S}.$$

Hence by Corollary 1.3, substituting a for b and c ,

$$(21) \quad \vdash_p \mathbf{F}(\mathbf{F}aa)(\mathbf{F}aa)(\mathbf{S}(A)).$$

Therefore by Corollary 1.3 again, substituting α for a ,

$$\vdash_p \mathbf{F}\alpha\alpha(\mathbf{S}(A))V_\alpha.$$

Now by Theorem 1 X has a p.t.s. α_0 , and α is an h.c.i. of α and α_0 , so by Corollary 1.3,

$$\vdash_p \alpha(\mathbf{S}(A))V_\alpha X.$$

It only remains to prove the existence of V_α for all α ; this is done by induction on the number of occurrences of \mathbf{F} in α (called the “length” of α).

Basis. If α is a variable, choose $V_\alpha \equiv \mathbf{I}$.

Induction step. Suppose α is $\mathbf{F}\beta\gamma$ for some β, γ . We shall construct V_α by induction on the number of variables common to β and γ .

Basis. Suppose there are no such variables. By the hypothesis of the induction on α we have

$$\vdash_p \mathbf{F}\beta\beta V_\beta, \quad \vdash_p \mathbf{F}\gamma\gamma V_\gamma.$$

Then using the C from Lemma 7, define

$$V_\alpha \equiv CV_\beta V_\gamma.$$

By Lemma 7, for new variables b and c ,

$$\vdash_p \mathbf{F}(\mathbf{F}bb)(\mathbf{F}(\mathbf{F}cc)(\mathbf{F}(\mathbf{F}bc)(\mathbf{F}bc)))C.$$

Hence by Corollary 1.3, substituting β for b ,

$$\vdash_p \mathbf{F}(\mathbf{F}cc)(\mathbf{F}(\mathbf{F}\beta c)(\mathbf{F}\beta c))(CV_\beta).$$

Now an h.c.i. of $\mathbf{F}cc$ and $\mathbf{F}\gamma\gamma$ is $\mathbf{F}\gamma\gamma$, so by Corollary 1.3 again, substituting γ for c ,

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)(CV_\beta V_\gamma)$$

as required. Notice that if a variable was common to β and γ , $\mathbf{F}\gamma\gamma$ would not satisfy the conditions for ξ^* in Corollary 1.3.

Induction step. Suppose there is a variable h which occurs in both β and γ . Let g be a new variable, and define

$$\beta' \equiv [g/h]\beta.$$

Then β' has one less variable in common with γ than β has, and

$$\beta \equiv [h/g]\beta'.$$

By the induction hypothesis there is a V such that

$$(22) \quad \vdash_p \mathbf{F}(\mathbf{F}\beta'\gamma)(\mathbf{F}\beta'\gamma)V.$$

We shall now construct a V_α such that

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)V_\alpha.$$

Case 1. When both β and γ are variables, then $\beta \equiv \gamma \equiv h$ and we need

$$\vdash_p \mathbf{F}(\mathbf{F}hh)(\mathbf{F}hh)V_\alpha.$$

Define V_α to be $\mathbf{S}(A\mathbf{I})$ and use (21).

Case 2. When β is compound, say $\beta \equiv \mathbf{F}\delta\epsilon$, and h occurs in δ . Then $\beta' \equiv \mathbf{F}\delta'\epsilon'$ and g occurs in δ' (but not in γ). Now $\mathbf{F}\delta\gamma$ is shorter than α because α is $\mathbf{F}(\mathbf{F}\delta\epsilon)\gamma$, so by the hypothesis of the induction on α there exists V_1 such that

$$\vdash_p \mathbf{F}(\mathbf{F}\delta\gamma)(\mathbf{F}\delta\gamma)V_1.$$

Also by (22),

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}\delta'\epsilon')\gamma)(\mathbf{F}(\mathbf{F}\delta'\epsilon')\gamma)V.$$

Now by Lemma 7, where ξ_1 is $\mathbf{F}(\mathbf{F}d\epsilon)c$ for new variables c, d, e ,

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_1\xi_1)(\mathbf{F}(\mathbf{F}(\mathbf{F}d\epsilon)(\mathbf{F}d\epsilon))(\mathbf{F}\xi_1\xi_1))B_1.$$

Define $V_\alpha \equiv B_1VV_1$.

Substituting $\delta', \epsilon', \gamma$ for d, e, c changes ξ_1 into $\mathbf{F}(\mathbf{F}\delta'\epsilon')\gamma$ which is $\mathbf{F}\beta'\gamma$. Hence by Corollary 1.3,

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}\delta'\gamma)(\mathbf{F}\delta'\gamma))(\mathbf{F}(\mathbf{F}\beta'\gamma)(\mathbf{F}\beta'\gamma))(B_1V).$$

Now $\delta \equiv [h/g]\delta'$ and g does not occur in γ , so $\mathbf{F}(\mathbf{F}\delta\gamma)(\mathbf{F}\delta\gamma)$ is an instance of $\mathbf{F}(\mathbf{F}\delta'\gamma)(\mathbf{F}\delta'\gamma)$. Also every variable in the former type-scheme occurs in the latter one (since h occurs in γ). Therefore Corollary 1.3 can be applied to B_1V and V_1 , with

$$\xi^* \equiv \zeta \equiv \mathbf{F}(\mathbf{F}\delta\gamma)(\mathbf{F}\delta\gamma),$$

$$\eta^* \equiv \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma).$$

Thus we obtain

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)(B_1VV_1)$$

as required.

Case 3. When β is compound, say $\beta \equiv \mathbf{F}\delta\epsilon$, and h occurs in ϵ but not in δ . Then β' is $\mathbf{F}\delta\epsilon'$ and g occurs in ϵ' . Now $\mathbf{F}\epsilon\gamma$ is shorter than α , so by hypothesis of the induction on α there exists V_2 such that

$$\vdash_p \mathbf{F}(\mathbf{F}\epsilon\gamma)(\mathbf{F}\epsilon\gamma)V_2.$$

Define $V_\alpha \equiv B_2VV_2$.

By Lemma 7, where ξ_2 is $\mathbf{F}(\mathbf{F}de)c$ for new variables d, e, c ,

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_2\xi_2)(\mathbf{F}(\mathbf{F}(Fec)(Fec))(\mathbf{F}\xi_2\xi_2))B_2.$$

Therefore by Corollary 1.3, substituting $\delta, \varepsilon', \gamma$ for d, e, c ,

$$\vdash_p \mathbf{F}(\mathbf{F}(F\varepsilon'\gamma)(F\varepsilon'\gamma))(\mathbf{F}(\mathbf{F}\beta'\gamma)(\mathbf{F}\beta'\gamma))(B_2V).$$

Similarly to Case 2, $\mathbf{F}(F\varepsilon\gamma)(F\varepsilon\gamma)$ is an instance of $\mathbf{F}(F\varepsilon'\gamma)(F\varepsilon'\gamma)$ and every variable in the former also occurs in the latter. Therefore by Corollary 1.3, substituting h for g ,

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)(B_2VV_2)$$

as required.

Case 4. When γ is compound, say $\gamma \equiv \mathbf{F}\delta\varepsilon$, with h occurring in δ . The scheme $\mathbf{F}\beta\delta$ is shorter than α , so we have a V_3 such that

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\delta)(\mathbf{F}\beta\delta)V_3.$$

Also by (22),

$$\vdash_p \mathbf{F}(\mathbf{F}\beta'(F\delta\varepsilon))(\mathbf{F}\beta'(F\delta\varepsilon))V.$$

Define $V_\alpha \equiv B_3VV_3$.

By Lemma 7, where ξ_3 is $\mathbf{F}b(\mathbf{F}de)$ for new variables b, d, e ,

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_3\xi_3)(\mathbf{F}(\mathbf{F}(Fbd)(Fbd))(\mathbf{F}\xi_3\xi_3))B_3.$$

Therefore by Corollary 1.3, substituting $\beta', \delta, \varepsilon$ for b, d, e ,

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}\beta'\delta)(\mathbf{F}\beta'\delta))(\mathbf{F}(\mathbf{F}\beta'\gamma)(\mathbf{F}\beta'\gamma))(B_3V).$$

And as in the previous cases we get

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)(B_3VV_3)$$

by substituting h for g and using Corollary 1.3.

Case 5. When γ is $\mathbf{F}\delta\varepsilon$ and h occurs in ε but not in δ . Then $\mathbf{F}\beta\varepsilon$ is shorter than α , so we can get a V_4 such that

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\varepsilon)(\mathbf{F}\beta\varepsilon)V_4.$$

Define $V_\alpha \equiv B_4VV_4$.

By Lemma 7, where ξ_4 is $\mathbf{F}b(\mathbf{F}de)$ for new variables b, d, e ,

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_4\xi_4)(\mathbf{F}(\mathbf{F}(Fbe)(Fbe))(\mathbf{F}\xi_4\xi_4))B_4.$$

As in Case 4, two applications of Corollary 1.3 give

$$\vdash_p \mathbf{F}(\mathbf{F}\beta\gamma)(\mathbf{F}\beta\gamma)(B_4VV_4),$$

completing the proof of Theorem 3.

COROLLARY 3.1. *The theorem remains true if α contains basic types, provided that to each basic type θ corresponds at least one ob Z_θ (not necessarily an atom) such that $\vdash_p \theta Z_\theta$.*

Proof. It is enough to prove the existence of a V_α for the case when α contains basic types. Let $\theta_1, \dots, \theta_n$ be the basic types in α , and let α_0 be the result of substituting a distinct new type-variable a_i for each θ_i in α . Then α will be the result of substituting $\theta_1, \dots, \theta_n$ for a_1, \dots, a_n in α_0 . By a nonprincipal deduction obtained by substituting α_0 for a in Example 1, §1,

$$\vdash \mathbf{F}\alpha_0\alpha_0\mathbf{I}.$$

Now by the axiom-scheme for \mathbf{K} ,

$$\vdash \mathbf{F}(\mathbf{F}\alpha_0\alpha_0)(\mathbf{F}a_n(\mathbf{F}\alpha_0\alpha_0))\mathbf{K},$$

so by Rule (F),

$$\vdash \mathbf{F}a_n(\mathbf{F}\alpha_0\alpha_0)(\mathbf{K}\mathbf{I}).$$

Repeating this argument, we get

$$\vdash (\mathbf{F}a_1 \cdots a_n(\mathbf{F}\alpha_0\alpha_0))(\mathbf{K}(\mathbf{K}(\cdots(\mathbf{K}\mathbf{I})\cdots))).$$

Therefore by Theorem 3 there exists a combinator U such that

$$\vdash_p (\mathbf{F}a_1 \cdots a_n(\mathbf{F}\alpha_0\alpha_0))U.$$

Now by the assumption, there exist obs Z_1, \dots, Z_n such that for each i ,

$$\vdash_p \theta_i Z_i.$$

Therefore by Corollary 1.3 used n times,

$$\vdash_p \mathbf{F}\alpha\alpha(UZ_1 \cdots Z_n),$$

so we can choose V_α to be $(UZ_1 \cdots Z_n)$, completing the proof.

REMARK. *Equality and types.* It might at first sight be hoped that if $X \geq Y$, then X and Y would have the same principal type-scheme. Theorem 3 shows that this is false, and the following example of Curry's,

$$\mathbf{SIII} \geq \mathbf{II}(\mathbf{II})$$

shows that X does not need to have a type at all, even though Y has types. All that we can say is that if $X \geq Y$ and X has types, then by (13), the p.t.s. of X is an instance of that of Y .

When $X = Y$ in general, the situation is even less tidy; in fact there exist such X and Y which have no type-schemes in common at all. For example, define

$$\alpha \equiv \mathbf{F}c(\mathbf{F}cc), \quad \beta \equiv \mathbf{F}(\mathbf{F}dd)d;$$

then by Theorem 3 there exist combinators I_1 and I_2 such that

$$\begin{aligned} I_1 &\geq \mathbf{I}, & I_2 &\geq \mathbf{I}, \\ \vdash_p \mathbf{F}\alpha\alpha I_1, & & \vdash_p \mathbf{F}\beta\beta I_2. \end{aligned}$$

Hence $I_1 = I_2$, but any type-scheme possessed by both I_1 and I_2 would have to have both the forms

$$\mathbf{F}(\mathbf{F}\gamma(\mathbf{F}\gamma\gamma))(\mathbf{F}\gamma(\mathbf{F}\gamma\gamma)), \quad \mathbf{F}(\mathbf{F}(\mathbf{F}\delta\delta)\delta)(\mathbf{F}(\mathbf{F}\delta\delta)\delta),$$

which is impossible.

Proof of Lemma 7. The following is one way of constructing the combinators B_1, \dots, B_4, C .

First define

$$D \equiv [x]. \mathbf{KI}(Ax).$$

Then

$$(23) \quad \vdash_p \mathbf{F}(\mathbf{Fab})(\mathbf{Fcc})D.$$

Proof. By Example 5, §4, we have

$$\vdash_p \mathbf{F}(\mathbf{Fab})(\mathbf{Fa}(\mathbf{Fab}))A.$$

A one-step principal deduction for x gives us

$$d x \vdash_p d x.$$

Now an h.c.i. of d and \mathbf{Fab} is \mathbf{Fab} itself, so by Corollary 1.4, substituting \mathbf{Fab} for d and combining the above two deductions gives

$$\mathbf{Fab} x \vdash_p \mathbf{Fa}(\mathbf{Fab})(Ax).$$

By the axiom-schemes and Example 3,

$$\vdash_p \mathbf{Fa}(\mathbf{Fda})\mathbf{K}, \quad \vdash_p \mathbf{Fcc}\mathbf{I},$$

so by Corollary 1.3, substituting \mathbf{Fcc} for a ,

$$(24) \quad \vdash_p \mathbf{Fd}(\mathbf{Fcc})(\mathbf{KI}).$$

An h.c.i. of d and $\mathbf{Fa}(\mathbf{Fab})$ is $\mathbf{Fa}(\mathbf{Fab})$, so by Corollary 1.4, substituting $\mathbf{Fa}(\mathbf{Fab})$ for d in (24),

$$\mathbf{Fab} x \vdash_p \mathbf{Fcc}(\mathbf{KI}(Ax)).$$

Finally Theorem 2 gives the result.

Next define $E \equiv [x, y]. D(A\mathbf{I}(\mathbf{Ky})x)y$. Then

$$(25) \quad \vdash_p \mathbf{F}(\mathbf{Fab})(\mathbf{Fbb})E.$$

Proof. First of all, we have

$$\vdash_p \mathbf{Fb}(\mathbf{Fab})\mathbf{K}, \\ b y \vdash_p b y.$$

Therefore by Corollary 1.5,

$$b y \vdash_p \mathbf{Fab}(\mathbf{Ky}).$$

By the start of the proof of Theorem 3,

$$(26) \quad \vdash_p \mathbf{Fc}(\mathbf{Fcc})(\mathbf{Al}).$$

Therefore by Corollary 1.4, substituting \mathbf{Fab} for c ,

$$b y \vdash_p \mathbf{F}(\mathbf{Fab})(\mathbf{Fab})(\mathbf{Al}(\mathbf{Ky})).$$

Now $c x \vdash_p c x$, so by Corollary 1.4, substituting \mathbf{Fab} for c ,

$$(27) \quad \mathbf{Fab} x, b y \vdash_p \mathbf{Fab}(\mathbf{Al}(\mathbf{Ky})x).$$

Therefore by (23) and Corollary 1.5,

$$\mathbf{Fab} x, b y \vdash_p \mathbf{Fcc}(D(\mathbf{Al}(\mathbf{Ky})x)).$$

Now for a new variable e ,

$$e y \vdash_p e y.$$

To combine the above two deductions by the proof of Theorem 1 we need an h.c.i. of the two sequences

$$\langle b, \mathbf{Fcc} \rangle \quad \langle e, \mathbf{Feg} \rangle.$$

A suitable h.c.i. is $\langle b, \mathbf{Fbb} \rangle$, so we can substitute b for c and e in the two deductions and combine them to get

$$\mathbf{Fab} x, b y \vdash_p b(D(\mathbf{Al}(\mathbf{Ky})x)y).$$

The result follows by Corollary 2.1.

Define $G \equiv [x, y]. \mathbf{Al}(\mathbf{Ky})x$. Then Corollary 2.1 applied to (27) gives us

$$(28) \quad \vdash_p \mathbf{F}(\mathbf{Fab})(\mathbf{Fb}(\mathbf{Fab}))G.$$

Define $H \equiv [u, x, y, z]. \mathbf{K}(\mathbf{Kz})(G(\mathbf{SK})yx)(Ex(uz))$. Then

$$(29) \quad \vdash_p \mathbf{F}(\mathbf{Fac})(\mathbf{F}(\mathbf{Fbc})(\mathbf{F}(\mathbf{Fbb})(\mathbf{Faa})))H.$$

Proof. By the proof of Example 3, §3,

$$\vdash_p \mathbf{F}(\mathbf{Fbc})(\mathbf{Fbb})(\mathbf{SK}).$$

Therefore by Corollary 1.3, substituting \mathbf{Fbc} for a and \mathbf{Fbb} for b in (28),

$$\vdash_p \mathbf{F}(\mathbf{Fbb})(\mathbf{F}(\mathbf{Fbc})(\mathbf{Fbb}))(G(\mathbf{SK})).$$

Now $d y \vdash_p d y$. Therefore by Corollary 1.4, substituting \mathbf{Fbb} for d ,

$$\mathbf{Fbb} y \vdash_p \mathbf{F}(\mathbf{Fbc})(\mathbf{Fbb})(G(\mathbf{SK})y).$$

Similarly,

$$(30) \quad \mathbf{Fbb} y, \mathbf{Fbc} x \vdash_p \mathbf{Fbb}(G(\mathbf{SK})yx).$$

Now by (25),

$$\vdash_p \mathbf{F}(\mathbf{Fbc})(\mathbf{Fcc})E.$$

Therefore by Corollary 1.4,

$$\mathbf{Fbc} x \vdash_p \mathbf{Fcc}(Ex).$$

Also $\mathbf{Fac} u, a z \vdash_p c(uz)$. Therefore by Corollary 1.5,

$$(31) \quad \mathbf{Fbc} x, \mathbf{Fac} u, a z \vdash_p c(Ex(uz)).$$

Now for new variables a, d, e , we have

$$\begin{aligned} &\vdash_p \mathbf{Fa}(\mathbf{Fda})\mathbf{K}, \\ &a z \vdash_p a z. \end{aligned}$$

Therefore by Corollary 1.5,

$$a z \vdash_p \mathbf{Fda}(\mathbf{K}z).$$

Also $\vdash_p \mathbf{Fb}(\mathbf{Feb})\mathbf{K}$. Therefore by Corollary 1.4, substituting \mathbf{Fda} for b ,

$$a z \vdash_p \mathbf{Fe}(\mathbf{Fda})(\mathbf{K}(\mathbf{K}z)).$$

To combine this with (30) by the proof of Theorem 1 we need an h.c.i. of

$$\mathbf{Fe}(\mathbf{Fda}), \quad \mathbf{F}(\mathbf{Fbb})g.$$

A suitable h.c.i. is $\mathbf{F}(\mathbf{Fbb})(\mathbf{Fda})$, since all its variables occur in \mathbf{Fbb} or in $\mathbf{Fe}(\mathbf{Fda})$; then substituting \mathbf{Fbb} for e in the above deduction and combining with (30) gives

$$\mathbf{Fbb} y, \mathbf{Fbc} x, a z \vdash_p \mathbf{Fda}(\mathbf{K}(\mathbf{K}z))(G(\mathbf{SK})yx)).$$

To combine this with (31) by the proof of Theorem 1 we need an h.c.i. of

$$\langle \mathbf{Fbc}, a, \mathbf{Fda} \rangle \quad \text{and} \quad \langle \mathbf{Fbc}, a, \mathbf{Fcg} \rangle.$$

A suitable h.c.i. is obtained by substituting c for d and a for g ; making the same substitution in the above deduction and combining with (31) gives

$$\mathbf{Fac} u, \mathbf{Fbb} y, \mathbf{Fbc} x, a z \vdash_p a(\mathbf{K}(\mathbf{K}z))(G(\mathbf{SK})yx)(Ex(uz)).$$

The result follows by Corollary 2.1.

Now suppose that for each $i = 1, \dots, 4$ there exist two combinators L_i and M_i and a type-scheme γ_i such that

$$\vdash_p \mathbf{F}\xi_i\gamma_i L_i, \quad \vdash_p \mathbf{F}\eta_i\gamma_i M_i$$

and γ_i contains all the variables in η_i . Then we can define

$$B_i \equiv G(HL_i M_i).$$

This has the required p.t.s., namely

$$\vdash_p \mathbf{F}(\mathbf{F}\xi_i\xi_i)(\mathbf{F}(\mathbf{F}\eta_i\eta_i)(\mathbf{F}\xi_i\xi_i))B_i.$$

Proof. By Corollary 1.3, substituting ξ_i for a and γ_i for c in (29) and assuming that the b therein is a new variable not occurring in ξ_i or γ_i

$$\vdash_p \mathbf{F}(\mathbf{F}b\gamma_i)(\mathbf{F}(\mathbf{F}bb)(\mathbf{F}\xi_i\xi_i))(HL_i).$$

Now an h.c.i. of $\mathbf{F}b\gamma_i$ and $\mathbf{F}\eta_i\gamma_i$ is $\mathbf{F}\eta_i\gamma_i$, and this satisfies the conditions of Corollary 1.3 since every variable in η_i is in γ_i , so by that corollary,

$$\vdash_p \mathbf{F}(\mathbf{F}\eta_i\gamma_i)(\mathbf{F}\xi_i\xi_i)(HL_iM_i).$$

Then the result follows by (28) and Corollary 1.3, substituting $\mathbf{F}\eta_i\gamma_i$ for a and $\mathbf{F}\xi_i\xi_i$ for b in (28).

Thus to construct B_1, \dots, B_4 , it only now remains to construct L_i and M_i for each i . When $i=1$: let γ_1 be $\mathbf{F}(\mathbf{F}de)(\mathbf{F}cc)$ and define

$$\begin{aligned} L_1 &\equiv [x, y, z]. A\mathbf{I}([u]. Duz)xy, \\ M_1 &\equiv [x, y]. E([z]. \mathbf{K}\mathbf{I}(yz)(xz)). \end{aligned}$$

Then

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}de)c)(\mathbf{F}(\mathbf{F}de)(\mathbf{F}cc))L_1$$

and

$$\vdash_p \mathbf{F}(\mathbf{F}dc)(\mathbf{F}(\mathbf{F}de)(\mathbf{F}cc))M_1.$$

Proof. For L_1 ; by (23),

$$\vdash_p \mathbf{F}(\mathbf{F}de)(\mathbf{F}cc)D.$$

Also $a u \vdash_p a u$. Therefore by Corollary 1.4, substituting $\mathbf{F}de$ for a ,

$$\mathbf{F}de u \vdash_p \mathbf{F}cc(Du).$$

Hence by Corollary 1.5,

$$c z, \mathbf{F}de u \vdash_p c(Duz).$$

Therefore by Theorem 2,

$$c z \vdash_p \mathbf{F}(\mathbf{F}de)c([u]. Duz).$$

Now by (26),

$$\vdash_p \mathbf{F}a(\mathbf{F}aa)(A\mathbf{I}),$$

so by Corollary 1.4, substituting $\mathbf{F}(\mathbf{F}de)c$ for a ,

$$c z \vdash_p \mathbf{F}(\mathbf{F}(\mathbf{F}de)c)(\mathbf{F}(\mathbf{F}de)c)(A\mathbf{I}([u]. Duz)).$$

Then using Corollary 1.4 twice we get

$$\mathbf{F}(\mathbf{F}de)c x, \mathbf{F}de y, c z \vdash_p c(A\mathbf{I}([u]. Duz)xy)$$

and the result for L_1 follows by Corollary 2.1.

For M_1 ; by (24) we have $\vdash_p \mathbf{F}e(\mathbf{F}cc)(\mathbf{K}\mathbf{I})$. Also

$$\mathbf{F}de y, d z \vdash_p e(yz).$$

Therefore by Corollary 1.5,

$$\mathbf{F}de y, d z \vdash_p \mathbf{F}cc(\mathbf{K}\mathbf{I}(yz)).$$

Similarly, by Corollary 1.5 again,

$$\mathbf{F}dc x, \mathbf{F}de y, d z \vdash_p c(\mathbf{K}\mathbf{I}(yz)(xz)).$$

Therefore by Theorem 2,

$$\mathbf{Fdc} x, \mathbf{Fde} y \vdash_p \mathbf{Fdc}([z]. \mathbf{KI}(yz)(xz)).$$

Now by (25),

$$\vdash_p \mathbf{F}(\mathbf{Fdc})(\mathbf{Fcc})E,$$

so by Corollary 1.5,

$$\mathbf{Fdc} x, \mathbf{Fde} y \vdash_p \mathbf{Fcc}(E([z]. \mathbf{KI}(yz)(xz)))$$

and the result for M_1 follows by Corollary 2.1.

When $i=2$; let γ_2 be $\mathbf{Fe}(\mathbf{Fcc})$ and define

$$L_2 \equiv [x, y]. \mathbf{AI}(x(\mathbf{Ky})),$$

$$M_2 \equiv [x, y]. \mathbf{AI}(xy).$$

Then

$$\vdash_p \mathbf{F}(\mathbf{F}(\mathbf{Fde})c)(\mathbf{Fe}(\mathbf{Fcc}))L_2$$

and

$$\vdash_p \mathbf{F}(\mathbf{Fec})(\mathbf{Fe}(\mathbf{Fcc}))M_2.$$

Proof. For L_2 ; we have

$$e y \vdash_p e y,$$

$$\vdash_p \mathbf{Fe}(\mathbf{Fde})\mathbf{K},$$

therefore by Corollary 1.5,

$$e y \vdash_p \mathbf{Fde}(\mathbf{Ky}).$$

Now

$$a x \vdash_p a x;$$

to combine these two deductions by the proof of Theorem 1 we need an h.c.i. of a and $\mathbf{F}(\mathbf{Fde})g$; since $\mathbf{F}(\mathbf{Fde})c$ itself is a suitable h.c.i., substituting $\mathbf{F}(\mathbf{Fde})c$ for a and applying Rule (F) gives us

$$\mathbf{F}(\mathbf{Fde})c x, e y \vdash_p c(x(\mathbf{Ky})).$$

Therefore by (26) and Corollary 1.5,

$$\mathbf{F}(\mathbf{Fde})c x, e y \vdash_p \mathbf{Fcc}(\mathbf{AI}(x(\mathbf{Ky})))$$

and the result for L_2 follows by Corollary 2.1.

For M_2 ; we have

$$\mathbf{Fec} x, e y \vdash_p c(xy).$$

Therefore by (26) and Corollary 1.5,

$$\mathbf{Fec} x, e y \vdash_p \mathbf{Fcc}(\mathbf{AI}(xy))$$

and the result follows by Corollary 2.1.

When $i=3$; let γ_3 be $\mathbf{Fb}(\mathbf{Fdd})$ and define

$$L_3 \equiv [x, y]. \mathbf{SK}(xy), \quad M_3 \equiv M_2.$$

Then

$$\vdash_p \mathbf{F}(\mathbf{Fb}(\mathbf{Fde}))(\mathbf{Fb}(\mathbf{Fdd}))L_3$$

and

$$\vdash_p \mathbf{F}(\mathbf{Fbd})(\mathbf{Fb}(\mathbf{Fdd}))M_3.$$

Proof. For M_3 , use the result for M_2 .

For L_3 ; by the proof of Example 3, §3,

$$\vdash_p \mathbf{F}(\mathbf{Fde})(\mathbf{Fdd})(\mathbf{SK}).$$

Also

$$\mathbf{Fbc} x, b y \vdash_p c(xy).$$

To combine these two deductions by the proof of Theorem 1, notice that $\mathbf{F}(\mathbf{Fde})(\mathbf{Fdd})$ is a suitable h.c.i. of $\mathbf{F}(\mathbf{Fde})(\mathbf{Fdd})$ and \mathbf{Fcg} , and substitute \mathbf{Fde} for c , giving

$$\mathbf{Fb}(\mathbf{Fde})x, b y \vdash_p \mathbf{Fdd}(\mathbf{SK}(xy)).$$

The result for L_3 follows by Corollary 2.1.

When $i=4$; let γ_4 be $\mathbf{Fb}(\mathbf{Fee})$ and define

$$L_4 \equiv [x, y].E(xy), \quad M_4 \equiv M_2.$$

Then

$$\vdash_p \mathbf{F}(\mathbf{Fb}(\mathbf{Fde}))(\mathbf{Fb}(\mathbf{Fee}))L_4$$

and

$$\vdash_p \mathbf{F}(\mathbf{Fbe})(\mathbf{Fb}(\mathbf{Fee}))M_4.$$

Proof. For L_4 ;

$$\mathbf{Fbc} x, b y \vdash_p c(xy)$$

and by (25),

$$\vdash_p \mathbf{F}(\mathbf{Fde})(\mathbf{Fee})E.$$

By the proof of Theorem 1, since $\mathbf{F}(\mathbf{Fde})(\mathbf{Fee})$ is a suitable h.c.i. of \mathbf{Fcg} and $\mathbf{F}(\mathbf{Fde})(\mathbf{Fee})$, we can substitute \mathbf{Fde} for c and combine the above two deductions to get

$$\mathbf{Fb}(\mathbf{Fde}) x, b y \vdash_p \mathbf{Fee}(E(xy)).$$

The result follows by Corollary 2.1.

The construction of B_1, \dots, B_4 is now complete, and only C remains to be found.

Define

$$C \equiv [u, x, y, z].Hxyu(yz).$$

Then

$$\vdash_p \mathbf{F}(\mathbf{Fbb})(\mathbf{F}(\mathbf{Fcc})(\mathbf{F}(\mathbf{Fbc})(\mathbf{Fbc})))C.$$

Proof. By (29) Corollary 1.4 used three times,

$$\mathbf{Fac} x, \mathbf{Fbc} y, \mathbf{Fbb} u \vdash_p \mathbf{Faa}(Hxyu).$$

Now

$$\mathbf{Fde} y, d z \vdash_p e(yz).$$

To combine these two deductions by the proof of Theorem 1 we need an h.c.i. of the sequences

$$\langle \mathbf{Fbc}, \mathbf{Faa} \rangle \quad \langle \mathbf{Fde}, \mathbf{Feg} \rangle.$$

A suitable h.c.i. is $\langle \mathbf{Fbc}, \mathbf{Fcc} \rangle$, so substituting c for e and a , and b for d and combining the deductions gives

$$\mathbf{Fcc} x, \mathbf{Fbc} y, \mathbf{Fbb} u, b z \vdash_p c(Hxyu(yz)).$$

Corollary 2.1 then gives the result, completing the proof of Lemma 7.

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